ON SOME DOUBLE-SERIES INEQUALITIES

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Abstract. We study certain double-series inequalities, which are motivated by weighted Hardy inequalities.

1. Introduction

Let $p>0$ and l^p be the space of all complex sequences $\mathbf{a}=(a_n)_{n\geq 1}$ satisfying:

$$\|\mathbf{a}\|_p=\Big(\sum_{i=1}^{\infty}|a_i|^p\Big)^{1/p}<\infty.$$

A matrix $A=(a_{n,k})$ is said to be a weighted mean matrix if its entries satisfy:

$$a_{n,k}=\lambda_k/\Lambda_n,\ 1\leq k\leq n;\quad \Lambda_n=\sum_{i=1}^{n}\lambda_i, \lambda_i\geq 0, \lambda_1>0.$$

For fixed $p>1$, the l^p operator norm of A is defined as the p-th root of the best possible constant U_p satisfying:

$$\sum_{n=1}^{\infty}\Big|\frac{1}{\Lambda_n}\sum_{k=1}^{n}\lambda_k a_k\Big|^p\leq U_p\sum_{n=1}^{\infty}|a_n|^p, \tag{1.1}$$

where the estimate is to hold for all complex sequences $\mathbf{a}$. When $\lambda_k=1$ for all k and $U_p=(p/(p-1))^p$, inequality (1.1) becomes the celebrated Hardy's inequality ([9, Theorem 326]).

By the duality principle [11, Lemma 2] for the norms of linear operators, inequality (1.1) is equivalent to the following

$$\sum_{n=1}^{\infty}\Big|\lambda_n\sum_{k=n}^{\infty}\frac{a_k}{\Lambda_k}\Big|^p\leq U_q^{p/q}\sum_{n=1}^{\infty}|a_n|^p, \tag{1.2}$$

where $q=p/(p-1)$.

From now on we restrict our attention to all non-negative sequences (a_n). Similar to inequality (1.2), one can also study the following inequality (or its reverse) for various p:

$$\sum_{n=1}^{\infty}\Big(\frac{1}{\Lambda_n}\sum_{k=n}^{\infty}\lambda_k a_k\Big)^p\leq U_p\sum_{n=1}^{\infty}a_n^p. \tag{1.3}$$

When $0<p<1$ and $\lambda_k=1$ for all k, the reversed inequality (1.3) becomes the one studied in Theorem 345 of [9]. The best possible constant U_p in this case is not yet known for all $0<p<1$. For studies in this direction, we refer the reader to the references [10, Theorem 61] and [8].

For fixed p, it is interesting to compare the right-hand side expressions in (1.2) and (1.3). When $\lambda_k=1$ for all k, one has the following result of Bennett and Grosse-Erdmann [4, Corollary 3]:

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\beta}}\sum_{k=n}^{\infty}a_k\Big)^p\leq\frac{1}{1-\beta p}\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{a_k}{k^{\beta}}\Big)^p.$$

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Here $0 < p \leq 1, 0 \leq \beta < 1/p$ and the constant is best possible.

More generally, for given matrices A, B, one can consider inequalities of the type

$$\|B\mathbf{x}\|_p \leq K(p)\|A\mathbf{x}\|_p, \tag{1.4}$$

where $K(p)$ is a constant, and the estimate is to hold for all non-negative sequences $\mathbf{x} = (x_n)$. When neither A nor B is a diagonal matrix, we refer to inequality (1.4) as double-series inequality. The double–series inequalities are studied in [4] and [3].

In this paper, we focus on the study of double–series inequalities given in the following form:

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} b_n c_k x_k\right)^p\right)^{1/p} \leq K(p,q)\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{n,k} x_k\right)^q\right)^{1/q}. \tag{1.5}$$

Here $(a_{n,k}), (b_n)$ and (c_n) are given non-negative sequences, $0 < p, q < \infty$ are fixed parameters. The estimate is to hold for all non-negative sequences $\mathbf{x}$. The constant $K(p,q)$ is independent of $\mathbf{x}$.

We denote $e^{(1)} = (1,0,0,\ldots), e^{(2)} = (0,1,0,\ldots)$ and so on. In Section 3, we prove the following

Theorem 1.1. *Suppose that $a_{n,k}/c_k$ increases with k for any fixed $n \geq 1$, then for $p \geq 1, 0 < q \leq p$, inequality (1.5) holds for non-negative sequences $x = (x_k)$ if and only if it holds for $x = e^{(n)} (n = 1, 2, \ldots)$. The theorem continues to hold when $0 < p \leq 1, q \geq p$ provided that inequality (1.5) is reversed.*

When $p = q$, a special case of Theorem 1.1 is proved in [4, Lemma 8] while the general case is proved in [3, Lemma 4].

Motivated by various choices for the λ_k's in (1.3), we apply Theorem 1.1 in Section 4 to determine the best possible constant $K(p)$ with $0 < p < 1$ in (1.4) for various A and B.

2. Lemmas

In this section we list a few lemmas that are needed in the proofs of our results in Section 4. We first note the following lemma of Levin and Stečkin [10, Lemma 1-2, p.18]:

Lemma 2.1. *For an integer $n \geq 1$,*

$$\sum_{i=1}^{n} i^r \geq \frac{1}{r+1} n(n+1)^r, \quad 0 \leq r \leq 1, \tag{2.1}$$

$$\sum_{i=1}^{n} i^r \geq \frac{r}{r+1}\frac{n^r(n+1)^r}{(n+1)^r - n^r}, \quad r \geq 1. \tag{2.2}$$

Inequality (2.1) reverses when $r \geq 1$ or $-1 < r \leq 0$. Inequality (2.2) reverses when $-1 < r \leq 1$.

Lemma 2.2. *For $s > r > -1$, $n \geq 1$,*

$$\frac{\sum_{i=1}^{n}(i/n)^r}{\sum_{i=1}^{n}(i/n)^s} < \frac{1+s}{1+r}. \tag{2.3}$$

The constant is best possible.

Proof. Upon letting $n \to \infty$, one sees easily that the constant is best possible. To prove inequality (2.3), we rewrite it as

$$f(n) := (1+s)\sum_{i=1}^{n} i^s - (1+r)n^{s-r}\sum_{i=1}^{n} i^r > 0.$$

Note that

$$f(n+1) - f(n) = (s-r)(n+1)^s - (1+r)\left((n+1)^{s-r} - n^{s-r}\right)\sum_{i=1}^{n} i^r.$$

We want to show the above expression is non-negative, which is amount to showing

$$\sum_{i=1}^{n} i^r \le \frac{(s-r)(n+1)^s}{(1+r)\left((n+1)^{s-r}-n^{s-r}\right)}.$$

For fixed r, it's easy to see that the right-hand side expression above is an increasing function of $s>r$ so that it suffices to show

$$\sum_{i=1}^{n} i^r \le \lim_{s\to r^+} \frac{(s-r)(n+1)^s}{(1+r)\left((n+1)^{s-r}-n^{s-r}\right)} = \frac{(n+1)^r}{(1+r)\ln(1+1/n)}. \tag{2.4}$$

As it's easy to see inequality (2.4) follows from various cases of inequalities (2.1) or (2.2), it follows that $f(n) \ge f(1) = s-r > 0$ for all $n \ge 1$ and this completes the proof. □

Lemma 2.3. *For $n \ge 2$,*

$$\frac{\sum_{i=1}^{n-1}(i/n)^r}{\sum_{i=1}^{n-1}(i/n)^s} \begin{cases} \le 2^{s-r} & \text{if } s>r\ge 1, \\ < \frac{1+s}{1+r} & \text{if } 0>s>r>-1. \end{cases}$$

The constants are best possible.

Proof. We first consider the case $s>r\ge 1$. Note that the case $n=2$ implies the constant here is best possible. To prove the corresponding inequality, we rewrite it as:

$$g(n) := 2^{s-r}\sum_{i=1}^{n-1} i^s - n^{s-r}\sum_{i=1}^{n-1} i^r > 0.$$

Note that

$$g(n+1)-g(n) = 2^{s-r}n^s - (n+1)^{s-r}n^r - \left((n+1)^{s-r}-n^{s-r}\right)\sum_{i=1}^{n-1} i^r.$$

We want to show the above expression is non-negative, which is amount to showing

$$\sum_{i=1}^{n-1} i^r \le \frac{2^{s-r}n^s-(n+1)^{s-r}n^r}{(n+1)^{s-r}-n^{s-r}} = n^r h\left(s-r; \frac{n}{n+1}\right), \tag{2.5}$$

where

$$h(u;v) = \frac{(2v)^u - 1}{1-v^u}.$$

Note that for $u>0, 1/2 \le v < 1$, we have

$$\frac{\partial h}{\partial u} = \frac{v^u}{(1-v^u)^2} p(u;\ln v).$$

where

$$p(u;w) = \left(2^u - (2e^w)^u\right)\ln 2 + w\left(2^u-1\right).$$

One sees easily that $p(u;w)$ is a concave function of w for fixed u and it follows that $p(u;\ln v) \ge \min(p(u;-\ln 2), p(u;0)) = 0$ for $1/2 \le v < 1$. We then deduce that in order to establish inequality (2.5) for $s>r\ge 1$, it suffices to show that

$$\sum_{i=1}^{n-1} i^r \le \lim_{s\to r^+} n^r h\left(s-r; \frac{n}{n+1}\right) = n^r\left(-1 + \frac{\ln 2}{\ln(1+\frac{1}{n})}\right).$$

As the above inequality is an easy consequence of the case $r\ge 1$ of inequality (2.1), we see that we have $g(n+1)-g(n) \ge 0$ for all $n\ge 2$ and $g(2)=0$, it follows that $g(n)\ge 0$ for all $n \ge 2$ and this completes the proof for the case $s>r\ge 1$.

Next, we consider the case $0 > s > r > -1$. Upon letting $n \to \infty$, one sees that the constant here is best possible. We prove the corresponding inequality by induction. When $n = 2$, the inequality follows easily from the fact that the function $r \mapsto (1+r)2^{-r}$ is an increasing function of $-1 < r < 0$.

Suppose now the corresponding inequality holds for some n with $n \geq 2$, then we have

$$\frac{\sum_{i=1}^{n}(i/(n+1))^r}{\sum_{i=1}^{n}(i/(n+1))^s} < (n+1)^{s-r}\frac{(\frac{1+s}{1+r})n^{r-s}\sum_{i=1}^{n-1} i^s + n^r}{\sum_{i=1}^{n} i^s}.$$

It suffices to show that the right-hand side expression above is $< \frac{1+s}{1+r}$, which is equivalent to the following

$$\sum_{i=1}^{n-1} i^s < \frac{n^s}{1+s}\left(-1-s+q\left(s-r;\frac{n}{n+1}\right)\right), \tag{2.6}$$

where

$$q(u;v) = \frac{u}{1-v^u}.$$

It's easy to see that for fixed $0 < v < 1$, $q(u;v)$ is an increasing function of $u > 0$. It follows that in order to establish inequality (2.6) for $0 > s > r > -1$, it suffices to show that

$$\sum_{i=1}^{n-1} i^s < \lim_{r \to s^-}\frac{n^s}{1+s}\left(-1-s+h\left(s-r;\frac{n}{n+1}\right)\right) = \frac{n^s}{1+s}\left(-1-s+\frac{1}{\ln(1+\frac{1}{n})}\right).$$

We now note the reversed inequality (2.1) valid for $-1 < r \leq 0$ implies that

$$\sum_{i=1}^{n-1} i^s \leq \frac{(n-1)n^s}{1+s}.$$

Thus, it remains to show that

$$\frac{(n-1)n^s}{1+s} < \frac{n^s}{1+s}\left(-1-s+\frac{1}{\ln(1+\frac{1}{n})}\right).$$

The above inequality is easily seen to be valid by noting that $-1 < s < 0$ and this completes the proof for the case $0 > s > r > -1$. $\square$

Lemma 2.4 ([7, Lemma 3.1])**.** *Let $\{B_n\}_{n=1}^{\infty}$ and $\{C_n\}_{n=1}^{\infty}$ be strictly increasing positive sequences with $B_1/B_2 \leq C_1/C_2$. If for any integer $n \geq 1$,*

$$\frac{B_{n+1}-B_n}{B_{n+2}-B_{n+1}} \leq \frac{C_{n+1}-C_n}{C_{n+2}-C_{n+1}}.$$

Then $B_n/B_{n+1} \leq C_n/C_{n+1}$ for any integer $n \geq 1$.

Lemma 2.5. *For $1 \leq s < r < 1/p$,*

$$\frac{\sum_{k=1}^{n}(r\sum_{i=1}^{k} i^{r-1})^{-p}}{\sum_{k=1}^{n}(s\sum_{i=1}^{k} i^{s-1})^{-p}} < \frac{1-sp}{1-rp}n^{(s-r)p}. \tag{2.7}$$

The constant $(1-sp)/(1-rp)$ is best possible.

Proof. We note first that as we have

$$k^r \leq r\sum_{i=1}^{k} i^{r-1} \leq (k+1)^r,$$

it's easy to see that the constant $(1-sp)/(1-rp)$ in (2.7) is best possible.

We now prove inequality (2.7) by induction. Note that when $n = 1$, this follows easily from the fact that the function $r \mapsto r^p/(1-rp)$ is an increasing function of $0 < r < 1/p$.

Suppose now inequality (2.7) holds for some n with $n \geq 1$, then we have

$$\frac{\sum_{k=1}^{n+1}(r\sum_{i=1}^{k} i^{r-1})^{-p}}{\sum_{k=1}^{n+1}(s\sum_{i=1}^{k} i^{s-1})^{-p}} < \frac{(\frac{1-sp}{1-rp})n^{(s-r)p}\sum_{k=1}^{n}(s\sum_{i=1}^{k} i^{s-1})^{-p} + (r\sum_{i=1}^{n+1} i^{r-1})^{-p}}{\sum_{k=1}^{n+1}(s\sum_{i=1}^{k} i^{s-1})^{-p}}.$$

It suffices to show that the right-hand side expression above is $< \frac{1-sp}{1-rp}(n+1)^{(s-r)p}$, which, after simplification, is equivalent to the following

$$\begin{aligned}
&(1-sp)\left(\left(1+\frac{1}{n}\right)^{(r-s)p}-1\right)\sum_{k=1}^{n}\left(s\sum_{i=1}^{k} i^{s-1}\right)^{-p} \\
<&(1-sp)\left(s\sum_{i=1}^{n+1} i^{s-1}\right)^{-p} - (1-rp)(n+1)^{-sp}\left(\frac{r}{n+1}\sum_{i=1}^{n+1}\left(\frac{i}{n+1}\right)^{r-1}\right)^{-p}.
\end{aligned}$$

We note that inequality (2.3) implies that for fixed $n \geq 1$, the function

$$r \mapsto (1+r)\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{r}$$

strictly increases with $r > -1$. It follows that we have

$$\begin{aligned}
&(1-sp)\left(s\sum_{i=1}^{n+1} i^{s-1}\right)^{-p} - (1-rp)(n+1)^{-sp}\left(\frac{r}{n+1}\sum_{i=1}^{n+1}\left(\frac{i}{n+1}\right)^{r-1}\right)^{-p} \\
>&(1-sp)\left(s\sum_{i=1}^{n+1} i^{s-1}\right)^{-p} - (1-rp)(n+1)^{-sp}\left(\frac{s}{n+1}\sum_{i=1}^{n+1}\left(\frac{i}{n+1}\right)^{s-1}\right)^{-p} \\
=&(r-s)p\left(s\sum_{i=1}^{n+1} i^{s-1}\right)^{-p}.
\end{aligned}$$

Thus, it remains to show that

$$\left(s\sum_{i=1}^{n+1} i^{s-1}\right)^{-p} \geq (1-sp)\frac{\left(1+\frac{1}{n}\right)^{(r-s)p}-1}{(r-s)p}\sum_{k=1}^{n}\left(s\sum_{i=1}^{k} i^{s-1}\right)^{-p}. \tag{2.8}$$

As it is easy to show that the function

$$x \mapsto \frac{\left(1+\frac{1}{n}\right)^{x}-1}{x}$$

is an increasing function for fixed n, it follows that we only need to establish inequality (2.8) with r replaced by $1/p$. After simplification, it is equivalent to the following inequality:

$$\frac{\sum_{k=1}^{n+1}\left(s\sum_{i=1}^{k} i^{s-1}\right)^{-p}}{\sum_{k=1}^{n}\left(s\sum_{i=1}^{k} i^{s-1}\right)^{-p}} \geq \frac{(n+1)^{1-sp}}{n^{1-sp}}. \tag{2.9}$$

In order to establish the above inequality, we first show that for any $n \geq 1$, we have

$$\frac{\sum_{k=1}^{n+1}\left(s\sum_{i=1}^{k} i^{s-1}\right)^{-p}}{\sum_{k=1}^{n}\left(s\sum_{i=1}^{k} i^{s-1}\right)^{-p}} \geq \frac{\sum_{k=1}^{n+1} i^{-sp}}{\sum_{k=1}^{n} i^{-sp}}.$$

The case $n=1$ of the above inequality can be easily established by observing that $s \geq 1$. We now apply Lemma 2.4 to conclude that it remains to show for any $n \geq 1$,

$$\frac{\left(\sum_{i=1}^{n+1} i^{s-1}\right)^{-p}}{\left(\sum_{i=1}^{n} i^{s-1}\right)^{-p}} \geq \frac{(n+1)^{-sp}}{n^{-sp}}.$$

The above inequality is equivalent to

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{i}{n+1}\right)^{s-1} \leq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{s-1}. \tag{2.10}$$

To establish the above inequality, we define for any function $f(x)$ defined on the interval $(0,1]$ and any integer $n \geq 1$,

$$R_n(f) = \frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n}).$$

Then a result [6, Theorem 3A] of Bennett and Jameson asserts that $R_n(f)$ decreases(resp. increases) with n if $f(x)$ is an increasing (resp. decreasing) function which is either convex or concave. This result applied to the function $f(x) = x^{s-1}$ leads immediately to inequality (2.10).

We now conclude that in order to show inequality (2.9), it remains to show that

$$\frac{\sum_{k=1}^{n+1} i^{-sp}}{\sum_{k=1}^{n} i^{-sp}} \geq \frac{(n+1)^{1-sp}}{n^{1-sp}}.$$

The above inequality is equivalent to

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{i}{n+1}\right)^{-sp} \geq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{-sp},$$

which also follows from the above mentioned result of Bennett and Jameson applied to $f(x) = x^{-sp}$. $\square$

3. Proof of Theorem 1.1

Motivated by the proof of [4, Lemma 8], we show that Theorem 1.1 is a consequence of the following

Theorem 3.1 ([2, Theorem 2], [5, Theorem 4])**.** *Let $0 < q \leq p < \infty$ and $p \geq 1$. Let $(a_{n,k})_{n,k\in\mathbb{N}}$ be a non-negative matrix, (b_k) be a non-negative sequence and let $C > 0$. Then*

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{n,k}x_k\right)^q\right)^{1/q} \geq C\left(\sum_{n=1}^{\infty} b_n x_n^p\right)^{1/p} \tag{3.1}$$

holds for all non-negative non-increasing sequences (x_n) if and only if for all $m \in \mathbb{N}$,

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{m} a_{n,k}\right)^q\right)^{1/q} \geq C\left(\sum_{n=1}^{m} b_n\right)^{1/p}. \tag{3.2}$$

The theorem continues to hold when $0 < p \leq q < \infty$ and $p \leq 1$ provided that inequalities (3.1) and (3.2) are reversed.

We may assume $p \geq 1, 0 < q \leq p$ here as the proof for the other case is similar. We denote $y_n = \sum_{k=n}^{\infty} c_k x_k, n \geq 1$ so that we have $y_1 \geq y_2 \geq \ldots \geq 0$, and that

$$x_n = \frac{y_n - y_{n+1}}{c_n}.$$

We can then recast inequality (1.5) as

$$\begin{aligned}\left(\sum_{n=1}^{\infty} b_n^p y_n^p\right)^{1/p} &\leq K(p,q)\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{n,k}\left(\frac{y_k-y_{k+1}}{c_k}\right)\right)^q\right)^{1/q} \\ &= K(p,q)\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\left(\frac{a_{n,k}}{c_k}-\frac{a_{n,k-1}}{c_{k-1}}\right)y_k\right)^q\right)^{1/q},\end{aligned}$$

where we set $a_{n,0}/c_0 = 0$. Note that by our assumption, $a_{n,k}/c_k$ increases with k for any fixed $n \geq 1$, so that

$$\frac{a_{n,k}}{c_k} - \frac{a_{n,k-1}}{c_{k-1}} \geq 0.$$

Now the assertion of Theorem 1.1 readily follows from Theorem 3.1.

4. Some Applications of Theorem 1.1

In this section we look at some applications of Theorem 1.1. All of our results in this section are motivated by (the reversed) inequality (1.3) for $0 < p < 1$. Thus we assume $0 < p < 1$ throughout this section and let $\mathbf{a} = (a_n)$ be any non-negative sequence. We first apply Theorem 1.1 with

$$a_{n,k} = \begin{cases} \frac{n^{-r}}{k^{1-r}} & \text{if } k \geq n, \\ 0 & \text{if } 1 \leq k < n \end{cases}, \ b_n = n^{-s}, \ c_k = 1/k^{1-s}, \ r > s$$

to see that

$$\sum_{n=1}^{\infty}\left(n^{-r}\sum_{k=n}^{\infty}\frac{a_k}{k^{1-r}}\right)^p \leq \sup_m \left(\frac{\sum_{n=1}^m (n/m)^{-rp}}{\sum_{n=1}^m (n/m)^{-sp}}\right)\sum_{n=1}^{\infty}\left(n^{-s}\sum_{k=n}^{\infty}\frac{a_k}{k^{1-s}}\right)^p.$$

It follows from Lemma 2.2 that we have the following

Theorem 4.1. *For $0 < p < 1$, $s < r < 1/p$, $a_n \geq 0$, we have*

$$\sum_{n=1}^{\infty}\left(n^{-s}\sum_{k=n}^{\infty}\frac{a_k}{k^{1-s}}\right)^p \leq \sum_{n=1}^{\infty}\left(n^{-r}\sum_{k=n}^{\infty}\frac{a_k}{k^{1-r}}\right)^p < \left(\frac{1-sp}{1-rp}\right)\sum_{n=1}^{\infty}\left(n^{-s}\sum_{k=n}^{\infty}\frac{a_k}{k^{1-s}}\right)^p. \tag{4.1}$$

The constants are best possible.

Note that the first inequality in (4.1) follows as we have plainly for $k \geq n, r > s$, $n^{-s}/k^{1-s} \leq n^{-r}/k^{1-r}$. Upon taking $a_1 = 1, a_k = 0, k \geq 2$, one sees that the first inequality in (4.1) is also best possible.

Next, we apply Theorem 1.1 with

$$a_{n,k} = \begin{cases} \frac{n^{-r}}{(k+1)^{1-r}} & \text{if } k \geq n, \\ 0 & \text{if } 1 \leq k < n \end{cases}, \ b_n = n^{-s}, \ c_k = 1/(k+1)^{1-s}, \ r > s$$

to see that

$$\sum_{n=1}^{\infty}\left(n^{-r}\sum_{k=n}^{\infty}\frac{a_k}{(k+1)^{1-r}}\right)^p \leq \sup_m \left(\frac{\sum_{n=1}^m (n/(m+1))^{-rp}}{\sum_{n=1}^m (n/(m+1))^{-sp}}\right)\sum_{n=1}^{\infty}\left(n^{-s}\sum_{k=n}^{\infty}\frac{a_k}{(k+1)^{1-s}}\right)^p.$$

It follows from Lemma 2.3 that we have the following

Theorem 4.2. *For $0 < p < 1$, $s < r < 1/p$, $a_n \geq 0$, we have*

$$\sum_{n=1}^{\infty}\left(n^{-r}\sum_{k=n}^{\infty}\frac{a_k}{(k+1)^{1-r}}\right)^p \leq C_{p,r,s}\sum_{n=1}^{\infty}\left(n^{-s}\sum_{k=n}^{\infty}\frac{a_k}{(k+1)^{1-s}}\right)^p,$$

where

$$C_{p,r,s}=\begin{cases} 2^{(r-s)p} & \text{if } s<r\leq -\frac{1}{p}, \\ \frac{1-sp}{1-rp} & \text{if } 0<s<r<\frac{1}{p}. \end{cases} \tag{4.2}$$

The constant $C_{p,r,s}$ is best possible.

Corollary 4.1. *Let $a_n \geq 0, 0<p<1$. For $0<\beta<\alpha<\frac{1}{p}$, we have*

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\alpha}}\sum_{k=n}^{\infty}\Big((k+1)^{\alpha}-k^{\alpha}\Big)a_k\Big)^p \leq \frac{\alpha^p}{\beta^p}C_{p,\alpha,\beta}\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\beta}}\sum_{k=n}^{\infty}\Big((k+1)^{\beta}-k^{\beta}\Big)a_k\Big)^p, \tag{4.3}$$

where $C_{p,\alpha,\beta}$ is defined as in (4.2) and the constant is best possible.

Proof. We apply Theorem 1.1 with

$$a_{n,k}=\begin{cases} n^{-\alpha}\left((k+1)^{\alpha}-k^{\alpha}\right) & \text{if } k\geq n, \\ 0 & \text{if } 1\leq k<n \end{cases}, \ b_n=n^{-\beta}, \ c_k=(k+1)^{\beta}-k^{\beta}, \ \alpha>\beta.$$

Note that the fact $a_{n,k}/c_k$ increases with k is an easy consequence of the Mean Value Theorem. Thus we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\alpha}}\sum_{k=n}^{\infty}\Big((k+1)^{\alpha}-k^{\alpha}\Big)a_k\Big)^p \\ \leq& \sup_m\left(\frac{\sum_{n=1}^m n^{-\alpha p}\left((m+1)^{\alpha}-m^{\alpha}\right)^p}{\sum_{n=1}^m n^{-\beta p}\left((m+1)^{\beta}-m^{\beta}\right)^p}\right)\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\beta}}\sum_{k=n}^{\infty}\Big((k+1)^{\beta}-k^{\beta}\Big)a_k\Big)^p.\end{aligned}$$

Note that by the Mean Value Theorem, we have

$$\frac{(m+1)^{\alpha}-m^{\alpha}}{(m+1)^{\beta}-m^{\beta}}=\frac{\alpha}{\beta}\xi^{\alpha-\beta}\leq\frac{\alpha}{\beta}(m+1)^{\alpha-\beta},$$

where $m<\xi<m+1$. It follows that

$$\sup_m\left(\frac{\sum_{n=1}^m n^{-\alpha p}\left((m+1)^{\alpha}-m^{\alpha}\right)^p}{\sum_{n=1}^m n^{-\beta p}\left((m+1)^{\beta}-m^{\beta}\right)^p}\right)\leq\frac{\alpha^p}{\beta^p}\sup_m\left(\frac{\sum_{n=1}^m (n/(m+1))^{-\alpha p}}{\sum_{n=1}^m (n/(m+1))^{-\beta p}}\right).$$

Inequality (4.3) then follows from Theorem 4.2. We further note that we have

$$\lim_{m\to+\infty}\frac{\sum_{n=1}^m n^{-\alpha p}\left((m+1)^{\alpha}-m^{\alpha}\right)^p}{\sum_{n=1}^m n^{-\beta p}\left((m+1)^{\beta}-m^{\beta}\right)^p}=\frac{\alpha^p}{\beta^p}C_{p,\alpha,\beta}.$$

This shows that the constant in (4.3) is best possible and this completes the proof. □

Upon letting $\beta\to 0^+$, we immediately obtain the following

Corollary 4.2. *Let $a_n \geq 0, 0<p<1$. For $0<\alpha<\frac{1}{p}$, we have*

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\alpha}}\sum_{k=n}^{\infty}\Big((k+1)^{\alpha}-k^{\alpha}\Big)a_k\Big)^p\leq\left(\frac{\alpha^p}{1-\alpha p}\right)\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\ln\left(\frac{k+1}{k}\right)a_k\Big)^p.$$

The constant is best possible.

Note that as $\ln(1+1/k)\leq 1/k$, we have the following

Corollary 4.3. *Let $a_n \geq 0, 0<p<1$. For $0<\alpha<\frac{1}{p}$, we have*

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\alpha}}\sum_{k=n}^{\infty}\Big((k+1)^{\alpha}-k^{\alpha}\Big)a_k\Big)^p\leq\left(\frac{\alpha^p}{1-\alpha p}\right)\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{a_k}{k}\Big)^p.$$

The constant is best possible.

We now consider an analogue of inequality (4.3) by taking

$$a_{n,k}=\begin{cases} k^{-\beta}\left(n^{\beta}-(n-1)^{\beta}\right) & \text{if } k\geq n, \\ 0 & \text{if } 1\leq k<n \end{cases}, \; b_n=n^{\alpha}-(n-1)^{\alpha},\; c_k=k^{-\alpha},\; \alpha>\beta.$$

Then it follows from Theorem 1.1 that

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\left(n^{\beta}-(n-1)^{\beta}\right)\sum_{k=n}^{\infty}k^{-\beta}a_k\right)^p \\ \leq &\sup_m\left(\frac{\sum_{n=1}^m(n^{\beta}-(n-1)^{\beta})^pm^{-\beta p}}{\sum_{n=1}^m(n^{\alpha}-(n-1)^{\alpha})^pm^{-\alpha p}}\right)\sum_{n=1}^{\infty}\left(\left(n^{\alpha}-(n-1)^{\alpha}\right)\sum_{k=n}^{\infty}k^{-\alpha}a_k\right)^p.\end{aligned}$$

We recall that for two positive real finite sequences $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)$, $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ if for all convex functions f, we have

$$\sum_{j=1}^n f(x_j)\leq\sum_{j=1}^n f(y_j).$$

We write $\mathbf{x}\leq_{maj}\mathbf{y}$ if this occurs and the majorization principle states that if (x_j) and (y_j) are decreasing, then $\mathbf{x}\leq_{maj}\mathbf{y}$ is equivalent to

$$\begin{aligned} x_1+x_2+\ldots+x_j &\leq y_1+y_2+\ldots+y_j \;(1\leq j\leq n-1), \\ x_1+x_2+\ldots+x_n &= y_1+y_2+\ldots+y_n.\end{aligned}$$

We refer the reader to [1, Sect. 1.30] for a simple proof of this.

Now suppose $0<\beta<\alpha\leq 1$, we apply the majorization principle to the convex function $-x^p$ and the two sequences

$$\mathbf{x}=\left(\frac{k^{\alpha}-(k-1)^{\alpha}}{n^{\alpha}}\right)_{1\leq k\leq n},\;\mathbf{y}=\left(\frac{k^{\beta}-(k-1)^{\beta}}{n^{\beta}}\right)_{1\leq k\leq n}.$$

It's easy to see that both $\mathbf{x}$ and $\mathbf{y}$ are decreasing and $\mathbf{x}\leq_{maj}\mathbf{y}$. It follows that

$$\frac{\sum_{n=1}^m(n^{\beta}-(n-1)^{\beta})^pm^{-\beta p}}{\sum_{n=1}^m(n^{\alpha}-(n-1)^{\alpha})^pm^{-\alpha p}}\leq 1.$$

As the above inequality becomes an identity when $m=1$, we obtain the following

Theorem 4.3. *Let $a_n\geq 0, 0<p<1$. For $0<\beta<\alpha\leq 1$, we have*

$$\sum_{n=1}^{\infty}\left(\left(n^{\beta}-(n-1)^{\beta}\right)\sum_{k=n}^{\infty}k^{-\beta}a_k\right)^p\leq\sum_{n=1}^{\infty}\left(\left(n^{\alpha}-(n-1)^{\alpha}\right)\sum_{k=n}^{\infty}k^{-\alpha}a_k\right)^p.$$

The constant is best possible.

Now, we apply Theorem 1.1 with

$$a_{n,k}=\begin{cases} \frac{k^{r-1}}{\sum_{i=1}^n i^{r-1}} & \text{if } k\geq n, \\ 0 & \text{if } 1\leq k<n \end{cases}, \; b_n=\left(\sum_{i=1}^n i^{s-1}\right)^{-1},\; c_k=k^{s-1},\; r>s$$

to see that

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\frac{1}{\sum_{i=1}^n i^{r-1}}\sum_{k=n}^{\infty}k^{r-1}a_k\right)^p \\ \leq &\sup_m\left(\frac{\sum_{n=1}^m(\sum_{i=1}^n i^{r-1})^{-p}m^{(r-1)p}}{\sum_{n=1}^m(\sum_{i=1}^n i^{s-1})^{-p}m^{(s-1)p}}\right)\sum_{n=1}^{\infty}\left(\frac{1}{\sum_{i=1}^n i^{s-1}}\sum_{k=n}^{\infty}k^{s-1}a_k\right)^p.\end{aligned}$$

It follows from Lemma 2.5 that we have the following

Theorem 4.4. *For $0 < p < 1$, $1 \leq s < r < 1/p$, $a_n \geq 0$, we have*
$$\sum_{n=1}^{\infty}\left(\frac{1}{\sum_{i=1}^n i^{r-1}}\sum_{k=n}^{\infty}k^{r-1}a_k\right)^p < \frac{r^p(1-sp)}{s^p(1-rp)}\sum_{n=1}^{\infty}\left(\frac{1}{\sum_{i=1}^n i^{s-1}}\sum_{k=n}^{\infty}k^{s-1}a_k\right)^p.$$
The constant is best possible.

We end this paper by considering an analogue to the above result. We apply Theorem 1.1 with
$$a_{n,k} = \begin{cases} \frac{n^{s-1}}{\sum_{i=1}^k i^{s-1}} & \text{if } k \geq n, \\ 0 & \text{if } 1 \leq k < n \end{cases}, \ b_n = n^{r-1}, \ c_k = \left(\sum_{i=1}^k i^{r-1}\right)^{-1}, \ r > s$$
to see that (note that the fact $a_{n,k}/c_k$ increases with k follows from a simple application of Lemma 2.4)
$$\sum_{n=1}^{\infty}\left(n^{s-1}\sum_{k=n}^{\infty}\frac{a_k}{\sum_{i=1}^n i^{s-1}}\right)^p \leq \sup_m\left(\frac{\sum_{n=1}^m n^{(s-1)p}(\sum_{i=1}^m i^{s-1})^{-p}}{\sum_{n=1}^m n^{(r-1)p}(\sum_{i=1}^m i^{r-1})^{-p}}\right)\sum_{n=1}^{\infty}\left(n^{r-1}\sum_{k=n}^{\infty}\frac{a_k}{\sum_{i=1}^n i^{r-1}}\right)^p.$$
Suppose now $s < r \leq 1$, we apply the majorization principle again to the convex function $-x^p$ and the two sequences
$$\mathbf{x} = \left(\frac{k^{r-1}}{\sum_{i=1}^n i^{r-1}}\right)_{1 \leq k \leq n}, \ \mathbf{y} = \left(\frac{k^{s-1}}{\sum_{i=1}^n i^{s-1}}\right)_{1 \leq k \leq n}.$$
It's easy to see that both $\mathbf{x}$ and $\mathbf{y}$ are decreasing and $\mathbf{x} \leq_{maj} \mathbf{y}$ (for example, by an application of Lemma 2.4). It follows that
$$\frac{\sum_{n=1}^m n^{(s-1)p}(\sum_{i=1}^m i^{s-1})^{-p}}{\sum_{n=1}^m n^{(r-1)p}(\sum_{i=1}^m i^{r-1})^{-p}} \leq 1.$$
As the above inequality becomes an identity when $m = 1$, we obtain the following

Theorem 4.5. *Let $a_n \geq 0, 0 < p < 1$. For $s < r \leq 1$, we have*
$$\sum_{n=1}^{\infty}\left(n^{s-1}\sum_{k=n}^{\infty}\frac{a_k}{\sum_{i=1}^n i^{s-1}}\right)^p \leq \sum_{n=1}^{\infty}\left(n^{r-1}\sum_{k=n}^{\infty}\frac{a_k}{\sum_{i=1}^n i^{r-1}}\right)^p.$$
The constant is best possible.

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